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A TURBULENT BURST MODEL OF WALL TURBULENCE FOR TWO-DIMENSIONAL TURBULENT BOUNDARY LAYER FLOW

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Abstract—Based on a review of modern approaches that have been developed for modeling wall turbulence, a model of the transport mechanism associated with the turbulent burst phenomenon is concluded to provide the most sound and practical basis for generalization. This basic approach is used to formulate a general model for momentum transfer within the wall region for two-dimensional turbulent boundary layer flow. Attention is also focused on practical implications of this general turbulent burst model and application is made to fully turbulent flow with small pressure gradients.

NOMENCLATURE

- f, Fanning friction factor;
- K, mean turbulent kinetic energy;
- p, pressure;
- $\bar{s}$, mean burst period;
- s^+, dimensionless mean burst period ($\equiv \bar{s}v/U^{*2}$);
- t, process time;
- u, axial component velocity distribution;
- u^+, dimensionless mean velocity distribution ($\equiv \bar{u}/U^*$);
- v, y-component velocity distribution;
- w, z-component velocity distribution;
- x, axial coordinate;
- y, distance from wall;
- z, spanwise coordinate.

Greek symbols

- ε, mean turbulent dissipation;
- ε_m, eddy viscosity;
- θ, age;
- ϕ, age distribution;
- ρ, density;
- $\bar{\tau}_t$, Reynolds stress.

Subscripts

- I, initial condition;
- M, interfacial condition.

Superscripts

- $\bar{\ }$, mean;
- $'$, fluctuating component.

INTRODUCTION

THE WALL region is of great practical and theoretical importance in wall-bounded turbulent flows. Flow visualization studies in steady two-dimensional turbulent boundary layers [1–8] have revealed that flow in this region consists of coherent vortex structures of low and high speed streaks alternating in the spanwise direction over the entire surface. These large scale elongated coherent structures have approximate streamwise and spanwise dimensionless mean dimensions of $\lambda_x^+ \simeq 440$ and $\lambda_z^+ \simeq 50$–100. The brief existence of such individual large scale elements within the wall region is associated with the turbulent burst process which includes both high axial velocity inrush and low axial velocity ejection phases. Because of the dynamic nature of the burst phenomenon, flow within the wall region in general and within individual coherent structures in particular is unsteady and three-dimensional in space, even though the mean flow field is only two-dimensional.

As a consequence of flow visualization, anemometry, and electrochemical studies of wall turbulence, use of the idea of a laminar sublayer in analyzing the wall region has been abandoned in recent years. The classical approach to analyzing transport in the wall region for steady two-dimensional turbulent boundary layer flows relies on the representation of turbulence characteristics in terms of mean and random fluctuating components and involves the solution of the time average continuity and momentum equations

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{1}$$

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right) \tag{2}$$

where $-\rho\overline{u'v'}$ is the important component of the mean turbulent shear stress tensor ($-\rho\overline{u_i'u_j'}$). The several classical approaches differ in how the mean turbulent shear stresses are specified. Three turbulence modeling approaches to developing inputs for turbulent stresses that are currently receiving attention include: (1) damping factor methods, (2) kinetic energy K transport equation methods, and (3) turbulent stress transport equation methods.

The first two of these methods employ the eddy viscosity ε_m concept in which the Reynolds stresses are assumed to be proportional to mean field gradients. For example, the pertinent mean turbulent stress $\bar{\tau}_t (\equiv -\rho\overline{u'v'})$ for two-dimensional turbulent boundary layer flow is given by

$$\bar{\tau}_t = \rho\varepsilon_m\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right) \simeq \rho\varepsilon_m\frac{\partial\bar{u}}{\partial y}. \quad (3)$$

Damping factor type models of wall turbulence have been undergoing a slow evolutionary process for more than two decades. In this approach, the unsteady one-dimensional (t, y) momentum equation is solved for the situation in which a plate oscillates longitudinally in a stationary fluid. The damping effect of the wall on ε_m is then inferred by discarding the unsteady terms in the solution for the instantaneous velocity distribution and by the use of several simplifying assumptions. The well known van Driest [9] damping factor approach has been adapted to several type flows with fair success [10–17]. As indicated by McEligot [18], the damping factor approach requires fewer adjustable constants than the transport equation models described by Launder and Spalding [19]. The damping factor formulation for eddy diffusivity ε_m is sometimes utilized in modern numerical methods for analyzing the fluid flow aspects of turbulent boundary layers. But the generality and usefulness of the damping factor approach is rather severely restricted because an artificial wall–fluid perspective is employed.

In the kinetic energy transport equation approach to modeling turbulence, transport equations are written for mean turbulent kinetic energy K and other mean turbulent transport characteristics. The two-equation K–ε transport equation approach is perhaps the most popular of the models of this type. (The transport equations for K and dissipation ε are obtained from the Navier–Stokes and continuity equations, and therefore have no independent fundamental character.) This approach has been applied with some success to the wall region [20–22]. For example, Jones and Launder [20, 21] have developed a two-equation K–ε model for transitional turbulent flow which involves the use of five empirical inputs. However, to account for the effects of low turbulent Reynolds number, rather arbitrary ad hoc assumptions (including damping factor corrections) have been employed in both the kinetic energy and the energy dissipation equations. As stated by the developers of these models, these assumptions for modeling wall turbulence still require further adjustments and refinements.

One-equation kinetic energy transport equation models have also been adapted to the wall region with some success [23–28]. However, the use of these models in the wall region generally involve van Driest type damping factor corrections for eddy viscosity. And the need to describe a length scale restricts the use of the one-equation model of turbulence to simple flow situations.

It should be noted that eddy viscosity methods such as the damping factor and kinetic energy transport equation models suffer from the sometimes serious defect that $\bar{\tau}_t$ does not always vanish at the same location at which $\partial\bar{u}/\partial y$ goes to zero. For example, $\bar{\tau}_t$ is known to be nonzero at the locations in the wall region at which a peak occurs in the mean velocity for turbulent wall jet flow and natural convection on a vertical surface. This behavior is not unusual for flows with asymmetrical profiles of mean quantities [29]. The underlying problem is that ε_m (and for that matter ε_K and ε_ε) are not trule scalar functions [30]. In an attempt to overcome this deficiency, mean turbulent stress transport equation models have been developed that avoid the use of equation (3). But this method generally involves the use of eddy diffusivity type assumptions for the transport of mean turbulent shear stress, and requires that the redistribution tensor and the dissipation tensor be modeled. To date, this task has not been satisfactorily accomplished for the wall region. Reynolds [31] and Launder [32] have indicated that it will be some time before models of this type are sufficiently well developed to be better than the simpler models for use in engineering analysis.

Because of uncertainties and limitations in the classical approaches that have been developed for modeling wall turbulence, standard numerical computation schemes currently available generally avoid numerical calculations in the wall region by utilizing simplifying wall functions. The lack of a general and reliable theoretical model of wall turbulence is one of the main limitations of existing numerical approaches for analyzing turbulent transport processes. The author of several of the computer programs that are widely used, Spalding, has suggested that the major work on modeling wall turbulence has yet to be done [33]. Spalding, Hubbard and Lightfoot [34]; Kays [35]; Lawn [36]; Liburdy *et al.* [37]; and others have sounded the call for a new style of thinking on this critical problem.

Alternative analysis approaches

Two alternative approaches to modeling wall turbulence which do not employ the classical equation for mean momentum have been put forth in recent years. Both of these approaches focus attention on the transport mechanism that is associated with the large scale coherent structure.

In the large eddy simulation approach, the unsteady three-dimensional Navier–Stokes equations for flow within a large scale eddy are transformed into large scale field equations by use of a statistical filter function. This spatial filtering precipitates additional unknowns in the form of Reynolds stresses and stress-like terms known as Leonard stresses. To solve the resulting large scale field equations, simplifying periodic boundary conditions are employed in the streamwise and spanwise directions and a traditional eddy viscosity model with related empirical inputs is used for the subgrid scale stresses. Based on modeling and empirical inputs for the initial disturbance and mean velocity profile, predictions have been developed for instantaneous velocity profiles, time averaged mean velocity profiles and turbulence statistics, and horizontally (xz plane) averaged turbulent quantities for fully turbulent, fully developed channel flow. This approach has been reported to characterize many of the important features of wall-bounded turbulent flows [38]. However, the formulation of the subgrid scale model for ε_m (or length scale) is not considered by the developers of this approach to be based on a well defined foundation. And, as already mentioned, the eddy viscosity concept has inerent weaknesses. The gap in computing time between the transport equation models and large eddy simulations is a large one, and it will be some time before this technique can be used for calculating flows of practical interest [31, 38].

A second alternative approach to analyzing turbulent transport within the wall region has evolved which treats wall turbulence as an unsteady transport phenomenon, without the use of classical eddy viscosity assumptions for $\overline{\tau_t}$, and without the need for developing higher order mean turbulent transport equations for kinetic energy, dissipation, or stresses. In this approach, unsteady transport of mass and momentum associated with the turbulent burst process is modeled, with a transformation into the mean domain being accomplished by the use of a statistical age distribution concept. The solution of the resulting mean transport equations for continuity and momentum gives rise to direct predictions for the mean velocity distribution within the wall region in terms of the mean frequency of the turbulent burst process $\bar{s}$.

This basic turbulent burst (or surface renewal) model has been adapted to standard wall bounded turbulent flow processes by Einstein and Li [39], Hanratty [40] and others (e.g. [41–48]). However, because this approach does not involve the use of the traditional eddy viscosity concept, surface renewal formulations for mean momentum transport within the wall region have essentially been developed outside the framework of the classical approaches to turbulence.

The surface renewal model of wall turbulence is somewhat similar to the van Driest [9] damping factor model. However, in this approach to modeling turbulent transport associated with the burst process, the fluid is taken as the fluctuating or intermittent medium with the relative velocity of the fluid at the wall appropriately set equal to zero, and the contribution of the unsteady fluctuating velocity distribution to the mean profile is accounted for statistically. As pointed out by Reynolds [49], the fundamental advantage of the surface renewal approach is its relation to a specific and not unrealistic picture of events in the viscous sublayer.

This basic approach will now be used to formulate a general model of the turbulent burst process for two-dimensional turbulent boundary layer flow, after which application will be made to fully turbulent flow with small pressure gradients.

GENERAL SURFACE RENEWAL MODEL FORMULATION

In the surface renewal approach to modeling wall turbulence, instantaneous transport equations are first written for the mass and momentum transfer associated with the life of an individual large scale coherent structure within the wall region. During the time between inrush and ejection over which the coherent structure resides within the wall region, the unsteady three-dimensional (t, x, y, z) flow field within the element is represented by the Navier–Stokes equations

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{4}$$

$$\frac{\partial u_i}{\partial \theta} + u_j \frac{\partial u_i}{\partial x_j} = \nu \frac{\partial^2 u_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial P}{\partial x_i} \tag{5}$$

and appropriate initial and boundary conditions; the age θ is equal to zero at the instant of inrush. [Equations of the form of equations (4) and (5) provide the starting point in the large scale eddy simulation approach, except that real process time t is used instead of the eddy age θ.]

Based on experimental data by Kim *et al.* [6], the inrush process can be assumed to be essentially instantaneous. Therefore, the important contribution of the inrush or surface renewal process to the transfer of momentum is accounted for by the initial condition

$$u_i = U_{Ii}, \quad \text{at } \theta = 0 \tag{6}$$

where U_{Ii} represents the velocity distribution at the instant of inrush.

Turning to the boundary conditions, the wall conditions and interfacial conditions between the wall region and the turbulent core can be modeled with reasonable confidence. These conditions are written as

$$u_i = 0, \quad \text{at } y = 0 \tag{7}$$

$$u_i = u_{Mi}, \quad \text{at } y = y_M \tag{8}$$

where u_{Mi} is time dependent. On the other hand, the unsteady streamwise and spanwise boundary conditions are complex and difficult to handle. It is these latter boundary conditions that are necessary to formally account for the interaction between adjacent large scale coherent structures.

According to flow visualization studies, the coherent structures of low and high speed large scale streaks alternate in the spanwise direction over the entire wall. At any instant of real (process) time t, the many large scale elements residing within the wall region can be assumed to have ages ranging from zero to relatively large values. (In general, the high speed elements will have small values of θ and the low speed coherent structures will have relatively larger values of θ.) The unsteady three-dimensional flow within the entire wall region at any instant t is therefore represented by equations of the form of equations (4)–(8), with the values of θ, U_{Ii}, u_{Mi}, and the streamwise and spanwise boundary conditions being dependent upon the his-

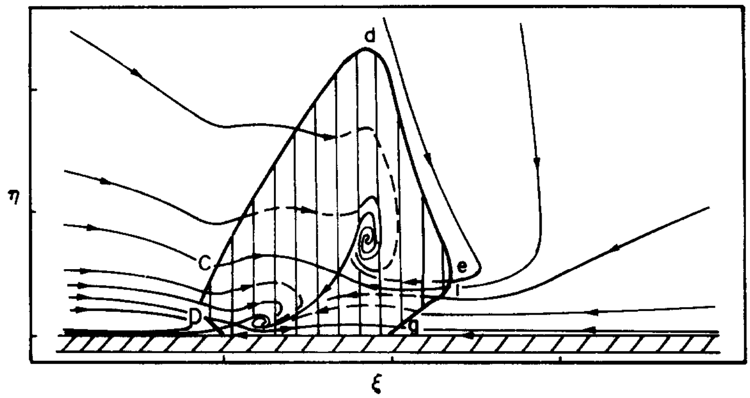

FIG. 1. Flow path in a single Emmons spot. The height of the spot is enlarged ten times. A Lagrangian coordinate system is used. After Cantwell *et al.* [58].

tory of each individual element.

The flow which is associated with this process is envisioned to involve an unsteady three-dimensional vortex pattern entrainment of fluid into the high speed large scale coherent structure from adjacent low speed areas. Consistent with this view, over the lifetime of an individual element within the wall region, fluid influx occurs for small values of θ and fluid efflux occurs for large values of θ. This general perspective is reinforced by the flow field anatomy of a single Emmons spot which is shown in Fig. 1. (The lines and arrows which indicate the paths of fluid particles entering the spot were obtained by Laser–Doppler anemometry and ensemble averaging.)

Recognizing that at any instant of time t the wall flow field consists of many interacting but coherent structures, the mean (ensemble average) distribution $\bar{\psi}$ in velocity and other characteristics over the entire surface is related to the instantaneous distribution ψ in each of a large number of samples by

$$\bar{\psi} = \int_{-\infty}^{\infty} \left[\int_0^{\infty} \psi\phi(\theta, \bar{s})\, d\theta \right] P_{\xi_i}(\xi_i)\, d\xi_i \tag{9}$$

where $\phi(\theta, \bar{s})$ is the statistical age distribution, $\bar{s}$ is the mean turbulent burst frequency, and $P_{\xi_i}(\xi_i)$ represents the probability distributions in the statistical initial distribution U_{Ii} and interfacial condition u_{Mi}. (According to the ergodic hypothesis, time t and ensemble averages are identical for stationary processes, such that $\bar{\psi}$ represents the time average distribution for the steady two-dimensional turbulent flow problem under consideration.)

The statistical age distribution is defined such that the product $\phi(\theta, \bar{s})\, d\theta$ represents the fraction of coherent structures with age between θ and $\theta + d\theta$. Furthermore, $\phi(\theta, \bar{s})$ must satisfy the equation

$$\int_0^{\infty} \phi(\theta, \bar{s})\, d\theta = 1. \tag{10}$$

Based on preliminary studies, predictions for $\bar{u}_i$ obtained on the basis of equation (9) have been found to be strongly dependent upon the magnitude of $\bar{s}$, but are fairly insensitive to the form of $\phi(\theta, \bar{s})$. Therefore, the convenient Danckwerts [50] exponential distribution is used, i.e.

$$\phi(\theta, \bar{s}) = \bar{s} \exp(-\theta\bar{s}). \tag{11}$$

The instantaneous equations given by equations (4)–(8) are transformed into the mean domain by the use of equations (9) and (10), with the result

$$\frac{d\overline{u_j}}{dx_j} = 0 \tag{12}$$

$$\bar{s}(\overline{u_i} - \overline{U_{Ii}}) + \overline{u_j \frac{\partial u_i}{\partial x_j}} = \nu \frac{\partial^2 \overline{u_i}}{\partial x_j} - \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} \tag{13}$$

and

$$\bar{u}_i = 0, \quad \text{at } y = 0 \tag{14}$$

$$\bar{u}_i = \overline{u_{Mi}}, \quad \text{at } y = y_M \tag{15}$$

where

$$\overline{U_{Ii}} = \int_{-\infty}^{\infty} U_{Ii} P_{U_{Ii}}(U_{Ii})\, dU_{Ii}. \tag{16}$$

Modeling approximations for $\overline{U_{Ii}}$ will be introduced later. The completion of a formal three-dimensional formulation would necessitate a transformation of appropriate streamwise and spanwise boundary conditions into the mean domain. (This complication was dealt with in the large scale eddy simulation work of Moin *et al.* [38] by the use of periodic boundary conditions at the sides of their computational box in the filtered large scale field.) However, pragmatic considerations to be put forth momentarily will provide a means of circumventing this problem for many practical cases.

Whereas the classical approaches and the large scale eddy simulation method introduce unknown Reynolds stress tensor terms in the transformed mean flow equations, the transformation of the fundamental instantaneous equations in the surface renewal approach introduces terms of the form $\bar{s}(\bar{u}_i - \overline{U_{Ii}})$ and $\overline{u_j\, \partial u_i/\partial x_j}$. The important eddy transport mechanism associated with the inrush process is accounted for by the term $\bar{s}(\bar{u}-\overline{U_{1i}})$. The term $\overline{u_j\,\partial u_i/\partial x_j}$ (together with the formal boundary conditions) represents the unsteady three-dimensional convective interaction between individual coherent structures and the surrounding fluid. The contribution of this complex convective vortex interaction to the establishment of the mean flow field is apparently minimized by the fact that the momentum influx to individual coherent structures over small values of age θ tends to be balanced by the momentum efflux during old age. Thus, it appears that the augmentation of mean transport by the turbulent burst process is primarily accounted for by the term $\bar{s}(\bar{u}_i-\overline{U_{1i}})$.

For steady two-dimensional turbulent boundary layer flow, $\bar{w}$ and all derivatives of mean characteristics with respect to z are zero. For this case, the mean flow field (i.e. $\bar{u}$, $\bar{v}$, $\partial\bar{P}/\partial y$) is represented by the continuity equation and the x- and y-component momentum equations

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}=0 \qquad (17)$$

$$\bar{s}(\bar{u}-\overline{U_{11}})+\overline{u\frac{\partial u}{\partial x}}+\overline{v\frac{\partial u}{\partial y}}+\overline{w\frac{\partial u}{\partial z}} = \nu\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}\right)-\frac{1}{\rho}\frac{\partial\bar{P}}{\partial x} \qquad (18)$$

$$\bar{s}(\bar{v}-\overline{U_{12}})+\overline{u\frac{\partial v}{\partial x}}+\overline{v\frac{\partial v}{\partial y}}+\overline{w\frac{\partial v}{\partial z}} = \nu\left(\frac{\partial^2\bar{v}}{\partial x^2}+\frac{\partial^2\bar{v}}{\partial y^2}\right)-\frac{1}{\rho}\frac{\partial\bar{P}}{\partial y} \qquad (19)$$

and accompanying boundary conditions. [Although equation (19) provides a theoretical basis for predicting $\partial P/\partial y$, this equation is not needed in practice since this term is known to be small.]

The z-component momentum equation reduces to

$$-\bar{s}\overline{U_{13}}+\overline{u\frac{\partial w}{\partial x}}+\overline{v\frac{\partial w}{\partial y}}+\overline{w\frac{\partial w}{\partial z}}=0. \qquad (20)$$

Based on the physics of the problem, the z- (and y-) component initial velocity distribution $\overline{U_{13}}$ (and $\overline{U_{12}}$) is much smaller than the axial component $\overline{U_{11}}$. With $\overline{U_{13}}$ assumed to be small, equation (20) indicates that the term $\overline{u_j\,\partial w/\partial x_j}$ is small, which reinforces the assumption that the effects of the three-dimensional convective interaction on the mean flow field are secondary.

Equations (17)–(19) and appropriate boundary conditions provide a theoretical basis for predicting the mean turbulent characteristic $\bar{u}$, $\bar{v}$, and $\partial\bar{P}/\partial y$. The formal solution of these equations in full form would first necessitate: (1) the specification of instantaneous streamwise and spanwise boundary conditions, (2) the solution of the instantaneous formulation [equations (4) and (5), initial conditions, and boundary conditions] for u, v, and w, and (3) a transformation to obtain $\overline{u_j\,\partial u/\partial x_j}$ and $\overline{u_j\,\partial v/\partial x_j}$. However, because the convective interaction terms appear to be secondary for many practical problems, equation (17)–(19) provide the basis for a simplified form of the surface renewal model that does not require the complex boundary condition formulation and computational effort associated with steps (1)–(3).

To provide a basis for rational simplification of the modeling equations, it is noted that as $\bar{s}$ approaches zero, the term $\overline{u_j\,\partial u_i/\partial x_j}$ clearly reduces to $\bar{u}_j\,\partial\bar{u}_i/\partial x_j$ and equations (17)–(19) reduce to the appropriate form for laminar conditions. Assuming that $\overline{u_j\,\partial u_i/\partial x_j}$ can be approximated by $\bar{u}_j\,\partial\bar{u}_i/\partial x_j$ and recognizing that $\partial^2\bar{u}/\partial x^2$ is generally very small, the surface renewal formulation for two-dimensional turbulent boundary layer flow reduces to [using equation (17)]

$$\bar{s}(\bar{u}-\overline{U_{11}})+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}=\nu\frac{\partial^2\bar{u}}{\partial y^2}-\frac{1}{\rho}\frac{\partial\bar{P}}{\partial x} \qquad (21)$$

$$\bar{u}=0, \quad \text{at } y=0 \qquad (22)$$

$$\bar{u}=\overline{u_{M1}}, \quad \text{at } y=y_M \qquad (23)$$

$$\bar{v}=0, \quad \text{at } y=0 \qquad (24)$$

and the single streamwise condition

$$\bar{u}=U_\infty, \quad \text{at } x=0. \qquad (25)$$

A practical application of this surface renewal model formulation is now briefly reviewed.

Application: fully turbulent boundary layer flow

For fully turbulent boundary layer flows with small pressure gradients, the effects of the convective terms and the pressure gradient term are insignificant within the wall region, such that the surface renewal formulation given by equations (17) and (21)–(25) reduces to the more manageable form

$$\bar{u}-\overline{U}_{11}=\nu\frac{d^2\bar{u}}{dy^2} \qquad (26)$$

with the boundary conditions given by equations (22) and (23). To close the model, $\overline{U_{11}}$, $\overline{u_{M1}}$, y_M, and $\bar{s}$ must be specified.

Similar simplified surface renewal model formulations that neglect the effects of convective interaction have been developed over the past few years for turbulent energy and momentum transfer in the wall region ([39]–[48] and others)†. These analyses differ primarily in how the initial condition $\overline{U}_{11}$ and the interfacial condition at y_M are handled.

According to flow visualization studies for fully turbulent internal flow by Popovich and Hummel [51], the inrush process carries fluid to within various small distances of the wall, with the mean approach distance H^+ approximately equal to 5.0. It follows that

† In most of these analyses, the transformation associated with equation (9) is made after the simplified instantaneous equations are solved for $u(\theta)$.

the initial profile U_{11} must be random and nonlinear. However, for practical purposes involving the development of first order predictions for momentum transfer, the modeling approximation

$$\overline{U_{11}} = U_{11} = U_{1c} \quad (27)$$

is often used, where U_{1c} is uniform. Models of this type developed by Einstein and Li [39]; Hanratty [40]; and others [41–48] lead to practical laws for u^+ in the wall region which are in basic agreement with experimental data. More comprehensive models which account for the effect of the unreplenished layer of fluid at the surface have been developed by Harriott [52]; Bullin and Duker [53]; and Thomas *et al.* [54, 55]. This type model, which is sometimes referred to as the surface rejuvenation model, provides a basis for developing higher order predictions for u^+ as well as predictions for the Reynolds stress, mixing length, and eddy diffusivity (or turbulent Prandtl number) very near the wall.

With respect to the interfacial conditions, most surface renewal model formulations published to date make use of the simplification

$$\bar{u} = \overline{U_{11}}, \quad \text{as } y \to \infty. \quad (28)$$

Model closure was accomplished in the early analyses of refs [39–47] by merely setting U_{11} equal to the bulk stream velocity U_b (or free stream velocity U_∞) and by specifying the friction factor, or by specifying u^+ at a point in the outer wall region. More recently [48], closure has been affected by interfacing the wall model with a classical eddy viscosity model for the turbulent core at the point at which continuity is maintained in u^+ and du^+/dy^+. But the use of equation (28) in the analyses of refs [39–48] gives rise to predictions for u^+ which must be truncated at y_M^+. To achieve a higher order interface that produces smooth and continuous predictions for u^+ throughout the entire inner region, the more general interfacial boundary condition is now employed.

The solution to equations (22), (23), and (26) with $\overline{U_{11}} = U_{1c}$ gives rise to an expression for the dimensionless mean velocity distribution u^+ ($\equiv \bar{u}/U^*$) of the form

$$u^+ = \{u_{M1}^+ \sinh(y^+\sqrt{s^+}) + U_{1c}^+ [\sinh(y_M^+\sqrt{s^+}) - \sinh(y^+\sqrt{s^+}) - \sinh((y_M^+ - y^+)\sqrt{s^+})]\} \times \frac{1}{\sinh(y_M^+\sqrt{s^+})} \quad (29)$$

where $s^+ = \bar{s}\nu/U^{*2}$. An independent relationship between s^+, U_{1c}^+, u_{M1}^+, and y_M^+ is obtained on the basis of the Newton law of viscous shear ($\bar{\tau}_0/\rho = U^{*2} = \nu\partial\bar{u}/\partial y|_0$)

$$\left.\frac{\partial u^+}{\partial y^+}\right|_0 = 1 \quad (30)$$

$$1 = \frac{\sqrt{s^+}}{\sinh(y_M^+\sqrt{s^+})}\{u_{M1}^+ + U_{1c}^+[\cosh(y_M^+\sqrt{s^+}) - 1]\}. \quad (31)$$

Three additional independent equations are written for s^+, U_{1c}^+, u_{M1}^+, and y_M^+ by requiring continuity in u^+, $\partial u^+/\partial y^+$, and $\partial^2 u^+/\partial y^{+2}$ at the interface between the wall region and the turbulent core. Based on the traditional mixing length model, the momentum equation in the overlap region for fully turbulent flow with small pressure gradient takes the form

$$\left(1 + l^{+2}\left|\frac{\partial u^+}{\partial y^+}\right|\right)\frac{\partial u^+}{\partial y^+} \simeq 1, \quad y/\delta \gtrsim 0.2 \quad (32)$$

where $l^+ \simeq \kappa y^+$. It follows that at the interface $y^+ = y_M^+$, u^+, $\partial u^+/\partial y^+$ and $\partial^2 u^+/\partial y^{+2}$ are given by

$$u^+ \simeq C + \frac{1}{\kappa}\ln y_M^+ \quad (33)$$

$$\frac{\partial u^+}{\partial y^+} \simeq \frac{1}{\kappa y_M^+} \quad (34)$$

$$\frac{\partial^2 u^+}{\partial y^{+2}} \simeq -\frac{1}{\kappa y_M^{+2}} \quad (35)$$

where $\kappa \simeq 0.41$ for boundary layer flows.

With s^+ specified on the basis of experimental data, the values of C, U_{1c}^+, u_{M1}^+, and y_M^+ for which u^+ satisfies equations (31) and (33)–(35) can be computed by algebraic elimination and iteration. Alternatively, with C specified, s^+, U_{1c}^+, u_{M1}^+, and y_M^+ can be computed. Following the second approach with C set equal to 5.0, the hydrodynamic modeling parameters are found to be $s^+ = 1/14.94^2$, $U_{1c}^+ = 14.93$, $u_M^+ = 14.73$, and $y_M^+ = 52.45$. This prediction for s^+ is shown in Fig. 2 to be compatible with experimental data for the mean period of the turbulent burst process for boundary layer flow with uniform free stream velocity and for fully developed tube flow.

The overall inner law for u^+ takes the form

$$u^+ = 14.93 - 0.01196 \sinh\left(\frac{y^+}{14.94}\right) - 0.8929 \sinh\left(\frac{52.45 - y^+}{14.94}\right), \quad y^+ \lesssim 52.45 \quad (36a)$$

$$u^+ = 5.0 + \frac{1}{0.41}\ln y^+, \quad y^+ \gtrsim 52.45. \quad (36b)$$

Equation (36) is compared with experimental data and the familar van Driest [9] equation in Fig. 3.

To complete the analysis, the inner laws can be coupled with equations for u^+ obtained by classical methods in the outer region. Predictions can then be obtained for the Fanning friction factor f.

As suggested earlier, more comprehensive surface renewal analyses have been developed which account for the effect of the unreplenished layer of fluid on $\overline{U_{11}}$. However, the simpler surface renewal analysis developed in this paper is judged to be quite adequate for analyzing turbulent momentum transfer in the wall region for boundary layer flows with small pressure gradients.

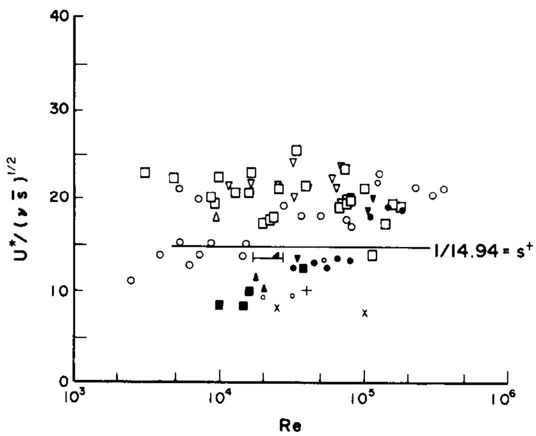

FIG. 2. Experimental data and theoretical predictions for mean period of turbulent burst. Reynolds number: $Re = D_H U_b/\nu$ for tube flow and $Re = 2\delta U_\infty/\nu$ for boundary layer flow.

Symbol	Type flow	Fluid	Measurement location y^+	Method	Ref.
⊢⊣	BL	water	15	visual	[59]
◢	BL	water	15	visual	[60]
▲	BL	air	wall	anemometer	[63]
●	BL	air	wall	anemometer	[64]
■	BL	air	>0	anemometer	[61]
▲	BL	water	>0	anemometer	[62]
°	TF	trichloro-ethylene	>0	visual	[7]
○	TF	air	wall	pressure	[45]
□	TF	air	wall	anemometer	[45]
△	TF	air	2	anemometer	[45]
▽	TF	tetraline	wall	anemometer	[45]
+	TF	air	7.56	anemometer	[65]
×	TF	water	>0	anemometer	[66]

TF, fully developed tube flow.
BL, boundary layer flow over flat plate.

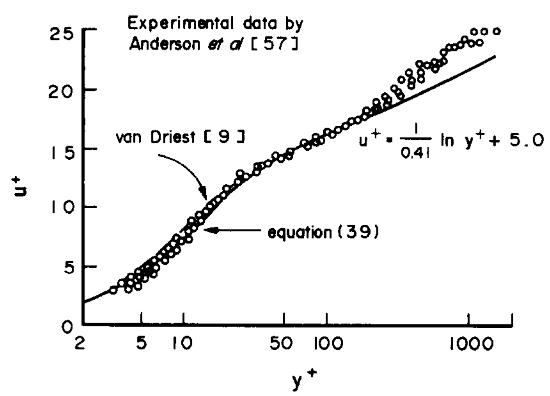

FIG. 3. Experimental data and inner laws for u^+ for turbulent boundary layer flow with uniform free stream velocity.

CONCLUSION

Of the models that have been developed for wall turbulence, the transport equation and large eddy simulation methods appear to be in the earliest stages of development, involve the highest degree of empiricism, and necessitate the greatest computer time and storage. The one-equation K transport equation model is the only one of these advanced approaches that have been sufficiently tested in the wall region, but even this method involves excessive computer costs [56]. Because of computational considerations and because of the complicated nonisotropic nature of turbulence in the wall region, it may be best to reserve the advanced turbulence models for use in the turbulent core.

The damping factor approach has been of great value in the early years of wall turbulence model development, but appears to be too artificial to be extended further.

The simplest, least empirical, and most computationally efficient of the approaches to modeling wall turbulence is the surface renewal method. This model of the turbulent burst phenomenon is felt to provide a practical approach to analyzing transport within the wall region. The analysis developed in this paper for fully turbulent boundary layer flow with small pressure gradients gives rise to a convenient analytical inner law for u^+ which is in excellent agreement with experimental data. When interfaced with a traditional mixing length representation for the turbulent core, this approach requires the specification of the single wall modeling parameter s^+. With the parameter C in the overlap law specified instead of s^+, the modeling predictions for the dimensionless mean burst frequency s^+ are compatible with experimental data.

The general surface renewal formulation developed in this paper is felt to provide a fundamental basis for generalization to account for the major effects of complicating factors such as strong adverse and favorable pressure gradients, transpiration, and heat and mass transfer. In this connection, the surface renewal approach provides a means of modeling the complex three-dimensional convective vortex interaction between large scale coherent structures having random phases in the wall region. The formal inclusion of the convective interaction term $\overline{u_j \, \partial u_i/\partial x_j}$ in the analysis will require the solution of the instantaneous formulation for u, v, and w and statistical transformation. Parenthetically, the term $\overline{u_j \, \partial u_i/\partial x_j}$ accounts for convection to coherent structures that are created by the inrush process but does not account for the eddy transport associated with the inrush process itself. [The inrush eddy transport mechanism is modeled by the term $\bar{s}(\bar{u} - \overline{U_{li}})$.] Because measurements for fluctuating components (particularly v') are dominated by the inrush process, the representation of $\overline{u_j \, \partial u_i/\partial x_j}$ in terms of the traditional Reynolds stress by the use of mean and fluctuating components would appear to have no physical basis.

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UN MODELE DE BOUFFEE TURBULENTE POUR LA TURBULENCE PARIETALE EN ECOULEMENT DE COUCHE LIMITE TURBULENTE ET BIDIMENSIONNELLE

Résumé—Basé sur une analyse des approches modernes qui ont été developpées pour modéliser la turbulence, un modèle de mécanisme de transport associé au phénomène de bouffée turbulente fournit une base pratique et la plus solide pour une généralisation. Cette approche est utilisée pour formuler un modèle général pour le transfert de quantité de mouvement dans la région pariétale d'une couche limite turbulente bidimensionnelle. L'attention est focalisée sur les implications pratiques de ce modèle général des bouffées et une application est faite à l'écoulement pleinement turbulent avec de faibles gradients de pression.

EIN MODELL DES TURBULENTEN BERSTENS DER WANDTURBULENZ FÜR ZWEIDIMENSIONALE TURBULENTE GRENZSCHICHTSTRÖMUNG

Zusammenfassung—Auf der Grundlage moderner Näherungsverfahren für die Berechnung der Wandturbulenz wurde ein Modell des Transportmechanismus, das mit dem Phänomen des turbulenten Berstens zusammenhängt, als die solideste und praktischste Basis für eine Verallgemeinerung ausgewählt. Hiervon ausgehend wird ein allgemeines Modell für die Impulsübertragung innerhalb der Wandregion zweidimensionaler turbulenter Grenzschichtströmungen formuliert. Dabei richtet sich das Interesse auch auf praktische Folgerungen aus diesem allgemeinen Modell des turbulenten Berstens, und es wird auf vollständig turbulente Strömung bei kleinen Druckgradienten angewandt.

МОДЕЛЬ ПРИСТЕННОЙ ТУРБУЛЕНТНОСТИ ПРИ ДВУМЕРНОМ ТУРБУЛЕНТНОМ ТЕЧЕНИИ В ПОГРАНИЧНОМ СЛОЕ, ОСНОВАННАЯ НА ЯВЛЕНИИ «ВЫБРОСА»

Аннотация — На основе проведенного обзора современных методов моделирования пристенной турбулентности разработана модель механизма переноса при турбулентном выбросе, позволяющая проводить точные и практически полезные обобщения. Метод использован для формулировки общей модели переноса импульса в пристенной области при двумерном турбулентном течении в пограничном слое. Особое внимание обращено на возможность практического использования предложенной модели турбулентного выброса. На ее основе проведен расчет полностью развитого турбулентного течения при небольших градиентах давления.